# Exact solution of an *N*-body problem in one dimension: two comments



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**COMMENT**

# Exact solution of an $N$-body problem in one dimension: two comments

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**Abstract.** We comment on a recent paper which presents two rather remarkable statements on the quantum-mechanical one-dimensional problem of $N$ equal mass particles interacting via the potential

$$V(\boldsymbol{x}) = g \sum_{j>k=1}^{N} (x_j - x_k)^{-2} + v(r) \qquad r \equiv \left[ N^{-1} \sum_{j>k=1}^{n} (x_j - x_k)^2 \right]^{1/2} \qquad v(r) = -\alpha'/r.$$

We point out that the first statement, concerning the bound-state spectrum, is the special case, corresponding to the 'Coulomb' potential $v(r) = -\alpha'/r$, of a more general result, valid for *any* 'radial' potential $v(r)$, while the second statement, concerning the scattering regimes, does not hold.

In a letter recently published in this journal [1] it was stated that the quantum-mechanical one-dimensional $N$-body problem characterized by the Hamiltonian

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + g \sum_{j>k=1}^{n} (x_j - x_k)^{-2} + v(r) \tag{1a}$$

$$r \equiv \left[ N^{-1} \sum_{j>k=1}^{N} (x_j - x_k)^2 \right]^{1/2} \tag{1b}$$

with

$$v(r) = -\alpha'/r \qquad \alpha' \equiv \alpha N^{-1/2} > 0 \tag{2}$$

has the following two remarkable properties: (i) its bound-state spectrum is given by the neat formula

$$E = -\tfrac{1}{4}\alpha'^2/(n+B)^2 \qquad n = 0, 1, 2, \ldots \tag{3}$$

with

$$B = \tfrac{1}{4}[N(N-1)(1+2g)^{1/2} + N(N+1) - 4] \tag{4}$$

and (ii) any incoming scattering configuration, characterized (in the sector of configuration space identified by the inequalities $x_j > x_{j+1}$, $j = 1, 2, \ldots, N-1$, to which consideration may be restricted without loss of generality) by *initial* momenta $p_j$ (of course, with

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$p_j < p_{j+1}$, $j = 1, 2, \ldots, N-1$), goes over into an outgoing scattered configuration *uniquely* defined by the *final* momenta $p'_j$ given by the simple rule

$$p'_j = p_{N+1-j} \qquad j = 1, 2, \ldots, N. \tag{5}$$

The purpose and scope of this comment is to point out that the (correct and remarkable) statement (i) is a special case of a more general proposition, while the statement (ii) does not hold (unless $\alpha'$, hence $v(r)$, vanishes).

The more general proposition that encompasses statement (i) reads as follows: the energy spectrum of the Hamiltonian (1) coincides (except for the multiplicities, which are, for simplicity, not discussed here; for a detailed treatment, see [2]) with the eigenvalues $E$ of the ODEs (see equations (2.11) or (4.4) of [2])

$$-\varphi''(r) - (2l+1)r^{-1}\varphi'(r) + v(r)\varphi(r) = E\varphi(r) \tag{6$a$}$$

or equivalently $(\psi(r) \equiv r^{l+\frac{1}{2}}\varphi(r))$, of the *radial Schrödinger equations*

$$-\psi''(r) + [l(l+1)r^{-2} + v(r)]\psi(r) = E\psi(r) \tag{6$b$}$$

with

$$l = B + k - 1 \qquad k = 0, 1, 2, \ldots \tag{6$c$}$$

and $B$ defined by (4).

This proposition is more general than statement (i) because it applies to *any* 'radial' potential $v(r)$, not merely to the 'Coulomb' potential (2). Its validity follows straightforwardly from the original treatment [2], on which the derivation of (3) is also based [1]. And the fact that this more general result entails (3) is rather trivial: it is a classical result of elementary quantum mechanics that (6*a*) or (6*b*) with (2) entail $E = -\frac{1}{4}\alpha'^2/q^2$ with $q = l + n_r$, $n_r = 1, 2, 3, \ldots$, namely, via (6*c*), precisely (3) (with $n = k + n_r - 1$).

Doubts about the validity of statement (ii) arise from equation (17) of [1], which displays a 'phase shift' $\eta_p$ dependent not only on the energy $E = p^2$ but also on the quantum number $k$, while the scattering wavefunction to which this phase shift is attached is not associated with a *single* value of the quantum number $k$, being rather a superposition of an infinity of eigenfunctions associated with all values $k = 0, 1, 2, \ldots$. In fact, it is not clear how equation (17) of [1] has been derived. The argument of [2], which is mentioned as having been followed in [1], requires that the quantities $c_{kq}$ be independent of $p$ (see equation (4.5) of [2], and the subsequent treatment); this is not so in the problem under consideration. (In fact, even the original treatment [2] can be faulted for having neglected this point; although in that case the matter is easily adjusted by a minor modification [3], thereby reconfirming the validity of the proof of (5) in the context of the problem treated in [2], namely (1) with $v(r) = 0$.)

This argument indicates that (5) has not been proven; this is not enough to exclude that (5) holds. A conclusive disproof of the validity of (5) for the Hamiltonian (1) with (2) and $\alpha' \neq 0$ is entailed by the following argument. If (5) were to hold in the quantum case, it would *a fortiori* hold in the classical case, and in particular in the three-body case $(N = 3)$. This would entail the formula

$$\varphi(+\infty) = \tfrac{1}{3}\pi - \varphi(-\infty) \qquad 0 < \varphi(\pm\infty) < \tfrac{\pi}{3} \tag{7}$$

where the quantity $\varphi(t)$ is the solution of the equations

$$\tfrac{1}{4}[r^2(t)\dot{\varphi}(t)]^2 = B^2 - \tfrac{9}{2}g\{\sin[3\varphi(t)]\}^{-2} \tag{8$a$}$$

$$\tfrac{1}{4}[\dot{r}(t)]^2 = E + \alpha' r^{-1} - B^2 r^2. \tag{8$b$}$$

For the validity of this assertion, and for the notation we employ (with $2m = 1$), the interested reader is referred to [4]. Note that formula (7) should follow from (8) for *all* (compatible) values of the two integration constants, $E > 0$ and $B^2 > \frac{9}{2}g$, and for *any* choice of $\varphi(-\infty)$ (in the interval $0 < \varphi(-\infty) < \pi/3$).

It is easily seen that equations (8) imply the relation

$$\int_{\varphi(-\infty)}^{\varphi(+\infty)} |1 - \tfrac{9}{2}gB^{-2}[\sin(3\varphi)]^{-2}|^{-\frac{1}{2}} = 2\int_{r_0}^{\infty} \mathrm{d}r\, r^{-1}(EB^{-2}r^2 + \alpha' B^{-2}r - 1)^{-1/2} \pmod{2\pi} \tag{9a}$$

$$r_0 = [(\alpha'^2 + 4EB^2)^{1/2} - \alpha']/(2E) \tag{9b}$$

namely, after some labour,

$$\varphi(+\infty) = \tfrac{\pi}{3} - \tfrac{1}{3}\arccos\{\cos[3\varphi(-\infty)] - \gamma\} \tag{10a}$$

$$\gamma = 2\beta(3-\beta^2)(1+\beta^2)^{-3/2}\{\cos[3\varphi(-\infty)] + \lambda^{-1}(1-3\beta^2)(1+\beta^2)^{-3/2}[1-\lambda^2\{\cos[3\varphi(-\infty)]\}^2]^{1/2}\} \tag{10b}$$

$$\beta = \alpha'/(2BE^{1/2}) \tag{10c}$$

$$\lambda = (1 - \tfrac{9}{2}gB^{-2})^{-1/2}. \tag{10d}$$

It is therefore clear that (7) holds for *all* (compatible) values of $E$, $B$ and $\varphi(-\infty)$ iff $\alpha' = 0$, entailing $\gamma = 0$.

Let us also point out that formula (37) of [5], which does imply the validity of (7) (since clearly $r(\pm\infty) = \infty$), and which might thereby have originated the idea that (5) holds [1], does not provide the most general trajectory entailed by (8), but presumably only a special case for which $\gamma$, see (10*b*), vanishes, hence (7) holds (note incidentally that $\gamma$ vanishes if $\beta^2 = \frac{1}{3}$, see (10*b*, *c*)).

In the first part of this comment we have emphasized the appropriateness of embedding the result (3), valid for the potential (2), in a more general context. Let us however end by also emphasizing that the discovery [1] of the neat formula (3) for the potential (2) indicates that this special case is indeed somewhat exceptional. Hence the possible solvability also in the scattering regime of the quantum-mechanical system (1) with (2), as well as the possible integrability of the corresponding classical Hamiltonian, emerge as intriguing open questions (the fact that the rule (5) does not hold does not entail that the corresponding classical Hamiltonian system is not integrable; indeed the systems under consideration are integrable for *any* $v(r)$ in the three-body case ($N = 3$) [4], and one might well wonder about their integrability for $N > 3$).

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